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Asymptotic Representation of Solutions of Equation $\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))]$

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In this paper the equation $\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))]$ is considered where $\tau \in C(I_{-1}, \mathbb{R}^+)$, $I_{-1} = [t_{-1}, \infty)$, $\mathbb{R}^+ = (0, \infty)$, $t - \tau(t)$ is an increasing function on I_{-1}, $\tau(t) \le \tilde{\tau}$, $t \in I_{-1}$, $0 < \tilde{\tau} = \text{const}$ and $\beta \in C(I_{-1}, \mathbb{R}^+)$. Results about asymptotic structure and behavior of the solutions of this equation are proved. A connection with the equation $\dot{x}(t) = -c(t)x(t - \tau(t))$ and the asymptotic representation of its solutions is obtained.

1. INTRODUCTION

In this paper, we consider the equation

$$\dot{y}(t) = \beta(t)[y(t) - y(t - \tau(t))], \tag{1}$$

where $\tau \in C(I_{-1}, \mathbb{R}^+)$, $I_{-1} = [t_{-1}, \infty)$, $t_{-1} \in \mathbb{R}$, $\mathbb{R}^+ = (0, \infty)$, $t - \tau(t)$ is an increasing function on I_{-1}, $\tau(t) \le \tilde{\tau}$, $t \in I_{-1}$, $0 < \tilde{\tau} = \text{const}$, and $\beta \in C(I_{-1}, \mathbb{R}^+)$, although in many affirmations, $\tau \in C(I, \mathbb{R}^+)$, $\beta \in C(I, \mathbb{R}^+)$, $I = [t_0, \infty)$, $t_{-1} = t_0 - \tau(t_0)$ is sufficient.

We prove results about asymptotic structure and behavior of the solutions of (1). A connection with the equation

$$\dot{x}(t) = -c(t)x(t - \tau(t)) \tag{2}$$

is shown and, as application of the obtained results, asymptotic representation of its solution is obtained.

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Equation (1) was investigated, e.g., in papers by F. V. Atkinson and J. R. Haddock [2] and S. N. Zhang [18]. Some questions related to the preceding problems are discussed, e.g., in the works by O. Arino, I. Györi, and M. Pituk, [1], R. Bellman and K. L. Cooke [3], K. Gopalsamy [9], I. Györi and M. Pituk [7, 8], E. Kozakiewicz [11] and T. Krisztin [12]. Therefore some comparisons to known results are made in the text of the paper.

2. PRELIMINARY

Consider the system of functional differential equations of retarded type,

$$\dot{y}(t) = f(t, y_t), \tag{3}$$

where $y \in \mathbb{R}^n$, y_t is an element from the space of continuous functions $C = C([-\tau, 0], \mathbb{R}^n)$, $y_t(\theta) = y(t+\theta)$ where $\theta \in [-\tau, 0]$, $f\colon \Omega \mapsto \mathbb{R}^n$, Ω is an open subset of $\mathbb{R} \times C$ and f is a continuous mapping such that the element $(\delta, \pi) \in \Omega$ determines a unique solution $y(\delta, \pi)$ on its maximal existence interval $D_{\delta, \pi} = [\delta, a)$, $\delta < a \le \infty$ (see, e.g., [10]). In the case of the linear equation (1) these conditions are satisfied and $a = \infty$.

In the paper, we use the topological principle of T. Ważewski [17] in the form of K. P. Rybakowski [16] for retarded functional differential equations. This modification applies the ideas of B. S. Razumikhin (see, e.g., [15]). Now we give a concept of this principle. (However, as usual, if $\omega \subset \mathbb{R} \times \mathbb{R}^n$, then int ω and $\partial\omega$ denote the interior and the boundary of ω, respectively.)

Definition 1 [16]. Let the continuously differentiable functions $l_i(t, y)$, $i = 1, 2, \dots, p$ and $m_j(t, y)$, $j = 1, 2, \dots, q$, $p^2 + q^2 > 0$ be defined on some open set $\omega_0 \subset \mathbb{R} \times \mathbb{R}^n$. The set,

$$\omega = \{(t, y) \in \omega_0 \colon l_i(t, y) < 0, m_j(t, y) < 0,$$
$$i = 1, 2, \dots, p, j = 1, 2, \dots, q\}$$

is called a *regular polyfacial set* with respect to the system (3) if (α) to (γ) in the following text hold:

(α) For $(t, \pi) \in \mathbb{R} \times C$ such that $(t + \theta, \pi(\theta)) \in \omega$ for $\theta \in [-\tau, 0)$, we have $(t, \pi) \in \Omega$.

(β) For all $i = 1, 2, \dots, p$, all $(t, y) \in \partial\omega$ for which $l_i(t, y) = 0$, and all $\pi \in C$ for which $\pi(0) = y$ and $(t + \theta, \pi(\theta)) \in \omega$, $\theta \in [-\tau, 0)$ it follows that $Dl_i > 0$ where,

$$Dl_i \equiv \frac{\partial l_i(t,y)}{\partial t} + \sum_{r=1}^{n} \frac{\partial l_i(t,y)}{\partial y_r} f_r(t,\pi).$$

(γ) For all $j = 1,2,\ldots,q$, all $(t,y) \in \partial\omega$ for which $m_j(t,y) = 0$, and all $\pi \in C$ for which $\pi(0) = y$ and $(t+\theta, \pi(\theta)) \in \omega$, $\theta \in [-\tau, 0)$ it follows that $Dm_j < 0$.

The elements $(t,\pi) \in \mathbb{R} \times C$ are in the sequel assumed to be such that $(t,\pi) \in \Omega$.

DEFINITION 2. A *system of initial functions* $p_{A,\omega}$ with respect to the nonempty sets A and ω where $A \subset \overline{\omega} \subset \mathbb{R} \times \mathbb{R}^n$ is defined as a continuous mapping p: $A \to C$ such that (α) and (β) in the following text hold:

(α) If $z = (t,y) \in A \cap \operatorname{int} \omega$ then $(t+\theta, p(z)(\theta)) \in \omega$ for $\theta \in [-\tau, 0]$.

(β) If $z = (t,y) \in A \cap \partial\omega$ then $(t+\theta, p(z)(\theta)) \in \omega$ for $\theta \in [-\tau, 0)$ and $(t, p(z)(0)) = z$.

The following lemma describes the main result of the paper by K. P. Rybakowski [16].

LEMMA 1. *Let $\omega \subset \omega_0$ be a regular polyfacial set with respect to the system* (3) *and let W be defined as follows,*

$$W = \{(t,y) \in \partial\omega \colon m_j(t,y) < 0,\ j = 1,2,\ldots,q\}.$$

Let $Z \subset W \cup \omega$ be a given set such that $Z \cap W$ is a retract of W but not a retract of Z. Then for each fixed system of initial functions $p_{Z,\omega}$ there is a point $z_0 = (\sigma_0, y_0) \in Z \cap \omega$ such that for the corresponding solution $y(\sigma_0, p(z_0))(t)$ of (3) *we have $(t, y(\sigma_0, p(z_0))(t)) \in \omega$ for each $t \in D_{\sigma_0, p(z_0)}$.*

3. SOME PROPERTIES OF SOLUTIONS

The solution $y = y(t)$ of (1) on interval I_{-1} is defined in a usual way as a function $y(t) \in C(I_{-1}, \mathbb{R}) \cap C^1(I, \mathbb{R})$ which satisfies (1) on I. Denote, as $y^{\varphi}(t)$, a solution of (1) to be generated by an *initial function* $\varphi(t)$, which is defined and continuous on an *initial interval* $[t_{-1}, t_0]$.

We start from following several lemmas. Obviously in view of the form of (1) the following lemma holds:

LEMMA 2. *If $y(t) = y^{\varphi}(t)$ is a solution of* (1) *on I_{-1} generated by an initial function $\varphi(t)$, then the function $Ky^{\varphi}(t) + L$ is a solution of* (1) *on I_{-1}, generated by initial function $K\varphi(t) + L$ for every constant $K, L \in \mathbb{R}$.*

LEMMA 3. *Let the initial function* $\varphi(t)$ *be defined and continuous on* $[t_{-1}, t_0]$ *and*

$$\varphi(t) < \varphi(t_0), \tag{4}$$

or

$$\varphi(t) > \varphi(t_0), \tag{5}$$

where $t \in [t_{-1}, t_0)$. *Then the corresponding solution* $y^{\varphi}(t)$ *is increasing in the case of inequality* (4) *or decreasing in the case of inequality* (5) *on* I.

Proof. Because it follows from (1) sign $\dot{y}^{\varphi}(t_0) = +1$ in the case (4) and sign $\dot{y}^{\varphi}(t_0) = -1$ in the case (5). The case $\dot{y}^{\varphi}(t^\star) = 0$ for a $t^\star \in (t_0, \infty)$ and sign $\dot{y}^{\varphi}(t) \neq 0$ where $t \in (t_0, t^\star)$ is impossible because, as follows from (1), $y(t^\star) \neq y(t^\star - \tau(t^\star))$. The lemma is proved.

LEMMA 4. *Let the initial function* $\varphi(t)$ *be continuous increasing positive* (*or decreasing negative*) *on* $[t_{-1}, t_0]$. *Then*

$$0 < |y^{\varphi}(t)| \leq |\varphi(t_0)| e^{\int_{t_0}^{t} \beta(s)\,ds}. \tag{6}$$

Proof. Let, e.g., $\varphi(t)$ be positive. By Lemma 3 $y^{\varphi}(t)$ increases on I. From representation,

$$y^{\varphi}(t) = e^{\int_{t_0}^{t} \beta(s)\,ds} \left[\varphi(t_0) - \int_{t_0}^{t} \beta(s) y^{\varphi}(s - \tau(s)) e^{-\int_{t_0}^{s} \beta(q)\,dq}\, ds \right], \qquad t \in I,$$

the inequality (6) follows. If $\varphi(t)$ is negative, we proceed by analogy. The lemma is proved.

4. REPRESENTATION OF SOLUTIONS

Consider the differential inequality,

$$\dot{\omega}(t) \leq \beta(t)[\omega(t) - \omega(t - \tau(t))], \tag{7}$$

where $t \in I$. The solution $\omega = \omega(t)$ of inequality (7) on interval $I_{-1} = [t_{-1}, \infty)$ is a function $\omega(t) \in C(I_{-1}, \mathbb{R}) \cap C^1(I, \mathbb{R})$ which satisfies the inequality (7) on I.

THEOREM 1. *If* $y(t)$ *is a solution of* (1) *on* I_{-1}, *then there is a solution* $\omega(t)$ *of inequality* (7) *on* I_{-1} *such that*

$$y(t) > \omega(t), \qquad t \in I_{-1}. \tag{8}$$

Conversely, if $\omega(t)$ *is a solution of inequality* (7) *on* I_{-1}, *then there is a solution* $y(t)$ *of* (1) *on* I_{-1} *such that* (8) *holds*.

Proof. If $y(t)$ is a solution of (1) on I_{-1}, then it is possible to put, e.g., $\omega(t) \equiv y(t) - 1$. Now suppose $\omega(t)$ is a solution of inequality (7) on I_{-1}. Let $\varphi(t)$ be an initial function on $[t_{-1}, t_0]$ such that $\varphi(t_0) = \omega(t_0)$ and $\varphi(t) < \omega(t)$ on $[t_{-1}, t_0)$. Prove that $y^{\varphi}(t) > \omega(t)$ on (t_0, ∞). Suppose the contrary. Then there exists a point $t^{\star} \in (t_0, \infty)$ such that $y^{\varphi}(t) > \omega(t)$ on $(t_0, t^{\star})$ and $y^{\varphi}(t^{\star}) = \omega(t^{\star})$. Note that the case $y^{\varphi}(t) \equiv \omega(t)$ on $[t_0, t^{\star})$ is impossible because in view of (7) for function $W(t) \equiv y^{\varphi}(t) - \omega(t)$ we have

$$\begin{aligned} \left.\frac{dW(t)}{dt}\right|_{t=t_0} &= \left[\beta(t)\left[y^{\varphi}(t) - y^{\varphi}(t - \tau(t))\right] - \dot{\omega}(t)\right]\Big|_{t=t_0} \\ &> \beta(t_0)\left[\omega(t_0) - \omega(t_0 - \tau(t_0))\right] - \dot{\omega}(t_0) \geq 0. \end{aligned}$$

Since $W(t_0) = W(t^{\star}) = 0$, then (in view of Rolle's theorem), there exists a point $t^{\star\star} \in (t_0, t^{\star})$ such that $W'(t) > 0$, $t \in [t_0, t^{\star})$ and $W'(t^{\star\star}) = 0$. This is impossible. Indeed, let a function $W_1(t) = W(t) - W(t^{\star\star})$ be involved. Then $W_1'(t^{\star\star}) = W'(t^{\star\star}) = 0$ and $y^{\varphi}(t) < \omega(t) + W(t^{\star\star})$ on $[t_{-1}, t^{\star\star})$. However, in view of (7),

$$\begin{aligned} \left.\frac{dW_1(t)}{dt}\right|_{t=t^{\star\star}} &= \left[\beta(t)\left[y^{\varphi}(t) - y^{\varphi}(t - \tau(t))\right] - \dot{\omega}(t)\right]\Big|_{t=t^{\star\star}} \\ &> \beta(t^{\star\star})\left[\omega(t^{\star\star}) + W(t^{\star\star})\right. \\ &\qquad \left. - \omega(t^{\star\star} - \tau(t^{\star\star})) - W(t^{\star\star})\right] \\ &\quad - \dot{\omega}(t^{\star\star}) \geq 0. \end{aligned}$$

This contradiction proves the theorem because (8) is valid, e.g., for $y(t) = y^{\varphi}(t) + 2\max_{[t_{-1}, t_0]}(\omega(t) - \varphi(t))$. From Theorem 1 immediately follows

THEOREM 2. *Equation* (1) *has a solution* $y^{\varphi}(t)$ *with property* $y^{\varphi}(\infty) = \pm\infty$ *iff inequality* (7) *has a solution* $\omega(t)$ *with property* $\omega(\infty) = \pm\infty$.

Remark 1. Inequality (7) has a solution $\omega = \omega(t)$ on I_{-1} of the form,

(i)

$$\omega(t) = k\int_{t_{-1}}^{t} \beta(s)\, ds + p, \qquad k \in \mathbb{R}^{+},\ p \in \mathbb{R},$$

if

$$\int_{t-\tau(t)}^{t} \beta(s)\,ds \geq 1, \qquad t \in I;$$

(ii)

$$\omega(t) = k\exp\left[\int_{t_{-1}}^{t} \varepsilon(s)\beta(s)\,ds\right] + p, \qquad k \in \mathbb{R}^+,\ p \in \mathbb{R},$$

if there is a function $\varepsilon(t) \in C(I_{-1}, \mathbb{R})$ such that

$$\varepsilon(t) \leq 1 - e^{-\int_{t-\tau(t)}^{t} \varepsilon(s)\beta(s)\,ds}, \qquad t \in I;$$

(iii)

$$\omega(t) = k\int_{t_{-1}}^{t}\left[\frac{1}{\tau(s)} - \beta(s)\right] ds + p, \qquad k \in \mathbb{R}^+,\ p \in \mathbb{R},$$

if

$$\frac{1}{\tau(t)\beta(t)} - 1 \leq \int_{t-\tau(t)}^{t}\left[\frac{1}{\tau(s)} - \beta(s)\right] ds, \qquad t \in I.$$

Solution $\omega(t)$ given by (i) obviously satisfies condition $\omega(+\infty) = +\infty$. Solution $\omega(t)$ given by (ii) satisfies condition $\omega(+\infty) = +\infty$ if, moreover,

$$\int^{+\infty} \varepsilon(s)\beta(s)\,ds = +\infty,$$

and in the case (iii) for $\omega(+\infty) = +\infty$ it is sufficient that

$$\int^{+\infty}\left[\frac{1}{\tau(s)} - \beta(s)\right] ds = +\infty.$$

THEOREM 3. *For each $C_1 \in \mathbb{R}$, $C_2 \in \mathbb{R}$, $C_1 < C_2$ there is a nonconstant solution $y = y(t)$ of* (1) *on I_{-1} such that*

$$C_1 < y(t) < C_2. \tag{9}$$

Proof. Let us show that the set $\omega = \{(t, y) \in \omega_0\colon m_1(t, y) < 0,\ l_1(t, y) < 0\}$ where $\omega_0 = (t_\varepsilon, \infty) \times \mathbb{R}$, $t_\varepsilon = t_0 - \varepsilon - \tau(t_0 - \varepsilon)$, $\varepsilon \in \mathbb{R}^+$, $t_{-1} + \varepsilon < t_0$, $m_1(t, y) = t_\varepsilon - t$ and $l_1(t, y) \equiv l_1(y) = (y - C_1)(y - C_2)$ is a regular polyfacial set with respect to (1). The condition (α) of Definition 1 is obviously fulfilled. Further $Dm_1 = -1 < 0$ and

$$Dl_1 = \beta(t)[2\pi(0) - C_1 - C_2][\pi(0) - \pi(-\tau(t))], \qquad t > t_0 - \varepsilon.$$

Since $C_1 < \pi(\theta) < C_2$ where $\theta \in [-\tau, 0)$, then

$$Dl_1|_{\pi(0)=C_1} = \beta(t)(C_1 - C_2)(C_1 - \pi(-\tau(t))) > 0, \qquad t > t_0 - \varepsilon,$$

and

$$Dl_1|_{\pi(0)=C_2} = \beta(t)(C_2 - C_1)(C_2 - \pi(-\tau(t))) > 0, \qquad t > t_0 - \varepsilon.$$

Now let us put $Z = \{(t, y) \in \overline{\omega},\ t = t_0\}$, $\Pi\colon W \ni (t^*, y^*) \mapsto (t_0, y^*) \in Z \cap W$ and choose a system of initial functions $p_{Z,\omega}$ which consists of *nonconstant* functions only. (This possibility follows from geometrical considerations.) All assumptions of Lemma 1 hold and then there is a point $z_0 = (t_0, y_0) \in Z \cap \omega$ such that the graph of the (nonconstant) solution which is defined by function $p(z_0) \in p_{Z,\omega}$ lies in ω for $t \in D_{t_0, p(z_0)} = [t_0, +\infty)$, that is its coordinates satisfy the inequalities (9). The theorem is proved.

THEOREM 4. *Let $Y(t)$ be a solution of* (1) *on I_{-1} with property $Y(+\infty) = +\infty$. Then for each solution $y(t)$ of* (1) *on I_{-1},*

$$y(t) = K \cdot Y(t) + \delta(t) \tag{10}$$

holds, where $K \in \mathbb{R}$ is a constant, depending on $y(t)$, and $\delta(t)$ is a bounded solution of (1) *on I_{-1} dependent on $y(t)$. This representation is unique* (*with respect to K and $\delta(t)$*).

Proof. Let $y(t)$ be bounded on I_{-1}. Then (10) holds for $K = 0$ and $\delta(t) \equiv y(t)$ only. Let $y(t)$ be not bounded. Without loss of generality, we can suppose that both $Y(t)$ and $y(t)$ are monotone on $[t_{-1}, t_0]$ (in opposite case both solutions become monotone in view of Lemma 3 on interval $[t_k, t_{k+1}]$ where $t_k = t_{k+1} - \tau(t_{k+1})$, $k \geq 0$ and the reasonings can be made on this interval), $y(t)$ is monotone decreasing and $Y(t) > y(t)$ on $[t_{-1}, t_0]$ (in view of Lemmas 2 and 3). Let $\varepsilon \in [0, 1]$ be a parameter. Consider the one-parametric family of functions,

$$\Phi(t, \varepsilon) = \varepsilon Y(t) + (1 - \varepsilon) y(t).$$

Obviously $\Phi(t, 1) = Y(t)$, $\Phi(t, 0) = y(t)$. In the proof of Theorem 3 we put the system $\Phi(t, \varepsilon)$, $\varepsilon \in [0, 1]$, $t \in [t_{-1}, t_0]$ as the system of initial functions $p_{Z,\omega}$. Put, moreover, $C_1 = y(t_0)$, $C_2 = Y(t_0)$, $C_1 < C_2$. By Theorem 3 there is a bounded solution $\tilde{\delta}(t)$ of (1) such that $C_1 < \tilde{\delta}(t) < C_2$ on I_{-1}. This solution corresponds to an initial function for a value of parameter

$\varepsilon = \tilde{\varepsilon} \in (0,1)$. Therefore,

$$\tilde{\delta}(t) = \tilde{\varepsilon} Y(t) + (1-\tilde{\varepsilon}) y(t), \qquad t \in I_{-1},$$

or

$$y(t) = \frac{\tilde{\varepsilon} Y(t)}{\tilde{\varepsilon}-1} + \frac{1}{1-\tilde{\varepsilon}} \tilde{\delta}(t), \qquad t \in I_{-1}.$$

Putting $K = \tilde{\varepsilon}/(\tilde{\varepsilon}-1)$, $\delta(t) = \tilde{\delta}(t)/(1-\tilde{\varepsilon})$, we obtain (10). If this representation is not unique, then for a $\varepsilon^{\diamond} \in (0,1)$, $\varepsilon^{\diamond} \neq \tilde{\varepsilon}$,

$$y(t) = \frac{\varepsilon^{\diamond} Y(t)}{\varepsilon^{\diamond}-1} + \frac{1}{1-\varepsilon^{\diamond}} \delta^{\diamond}(t), t \in I_{-1},$$

where $\delta^{\diamond}(t)$ is a solution of (1) bounded on I_{-1} and

$$\left(\frac{\tilde{\varepsilon}}{\tilde{\varepsilon}-1} - \frac{\varepsilon^{\diamond}}{\varepsilon^{\diamond}-1}\right) Y(t) = \frac{1}{1-\varepsilon^{\diamond}} \delta^{\diamond}(t) - \frac{1}{1-\tilde{\varepsilon}} \tilde{\delta}(t).$$

This is impossible because $\tilde{\varepsilon}(\tilde{\varepsilon}-1)^{-1} \neq \varepsilon^{\diamond}(\varepsilon^{\diamond}-1)^{-1}$ and $Y(+\infty) = +\infty$. The theorem is proved.

Remark 2. Obviously, for each $K \in \mathbb{R}$ and for each bounded solution $\delta(t)$ of (1) there is a solution $y(t)$ of (1) given by (10), i.e., the converse affirmation is valid. If Theorem 4 holds, then in the representation (10) the bounded solution $\delta(t)$ can be written in the form $\delta(t) = L + \delta_1(t)$ where $\delta_1(t)$ is a bounded solution of (1), $L \in \mathbb{R}$ and $\delta_1(t) = O(L)$. Then

$$y(t) = K \cdot Y(t) + L + \delta_1(t).$$

This representation is not unique.

5. EXPONENTIAL ASYMPTOTICS OF SOLUTIONS

LEMMA 5. *If $y(t)$ is a positive solution of* (1) *on I_{-1}, then the expression,*

$$V(t) = e^{-\int_{t-1}^{t} \beta(s)\, ds} \cdot y(t)$$

is a decreasing function on I.

Proof. For $V'(t)$ we have

$$\begin{aligned} V'(t) &= e^{-\int_{t-1}^{t} \beta(s)\, ds} [y'(t) - \beta(t) y(t)] \\ &= -e^{-\int_{t-1}^{t} \beta(s)\, ds} \beta(t) y(t-\tau(t)) < 0, \qquad t \in I. \end{aligned}$$

The lemma is proved.

THEOREM 5. *Solution $y = y(t)$, $t \in I_{-1}$ of* (1) *is represented in the form,*

$$y(t) = (\xi + \gamma(t))e^{\int_{t-1}^{t}\beta(s)\,ds}, \qquad t \in I_{-1}, \tag{11}$$

where ξ is a positive constant and $\gamma(t)$ is a continuous positive function on I with property $\gamma(+\infty) = 0$ iff there is a function $\alpha(t) \in C(I_{-1}, \mathbb{R}^+)$ such that $\int^{+\infty}\alpha(s)\beta(s)\,ds < \infty$ and

$$\alpha(t) \geq e^{-\int_{t-\tau(t)}^{t}(1-\alpha(s))\beta(s)\,ds}, \qquad t \in I. \tag{12}$$

Proof. A. By Theorem 1 and part (ii) of Remark 1, if $\varepsilon(t) \equiv 1 - \alpha(t)$ is put, there is a solution $y = y(t)$ of (1) such that

$$y(t) > \omega(t) = ke^{\int_{t-1}^{t}(1-\alpha(s))\beta(s)\,ds}, \qquad k \in \mathbb{R}^+, t \in I_{-1}. \tag{13}$$

By Lemma 5 there exists $\lim_{t\to\infty} V(t) = \xi \geq 0$, i.e.,

$$y(t) = (\xi + \gamma(t))e^{\int_{t-1}^{t}\beta(s)\,ds}, \qquad t \in I_{-1}, \tag{14}$$

where $\gamma(t)$ has properties indicated in formulation of theorem. Comparing (13) and (14) and supposing that $\xi = 0$, we have

$$\gamma(t)e^{\int_{t-1}^{t}\beta(s)\,ds} > ke^{\int_{t-1}^{t}(1-\alpha(s))\beta(s)\,ds},$$

or

$$\gamma(t) > ke^{-\int_{t-1}^{t}\alpha(s)\beta(s)\,ds}.$$

The last inequality cannot be valid because the limit of the right-hand side is nonzero for $t \to \infty$. Therefore $\xi \geq k > 0$.

B. Let $y(t)$ be of the form (11). Denote $\alpha_1(t) \equiv y(t - \tau(t))y^{-1}(t)$. From (1) we obtain

$$-\frac{\dot{y}(t)}{y(t)} = -\beta(t)(1 - \alpha_1(t)), \qquad t \in I,$$

and by integration from $t - \tau(t)$ to t we have

$$\alpha_1(t) \equiv e^{\int_{t-\tau(t)}^{t}(1-\alpha_1(s))\beta(s)\,ds}, \qquad t \in I_1 = [t_1, \infty),$$

that is (12) holds for $\alpha \equiv \alpha_1(t)$ on I_1. Moreover, from (1) follows

$$\int_{t_0}^{t}\left[\frac{\dot{y}(s)}{y(s)} - \beta(s)\right]ds = -\int_{t_0}^{t}\alpha_1(s)\beta(s)\,ds,$$

and, using (11), we have

$$\frac{\xi + \gamma(t)}{\xi + \gamma(t_0)} = e^{-\int_{t_0}^{t} \alpha_1(s)\beta(s)\, ds}.$$

From this follows the convergence of integral $\int^{+\infty} \alpha_1(s)\beta(s)\, ds$. Defining $\alpha(t - \tau(t)) \equiv \alpha_1(t)$ in (12) means the end of the proof.

Remark 3. The function $\alpha(t)$ often can be taken in the form $\alpha(t) = [\beta^{p+1}(t)]^{-1}$, $p \in \mathbb{R}^+$ if $\int^{\infty} \beta^{-p}(s)\, ds < \infty$. E.g., if $\beta(t) = t$ and $\tau(t) \in [\tau_1, \tilde{\tau}]$, $0 < \tau_1 =$ const, then it is possible to put (if t_{-1} is sufficiently large) $\alpha(t) = t^{-3}$.

THEOREM 6. *Let there be a solution $y(t)$ of* (1) *in the form* (11) *on I_{-1} and*

$$\int^{\infty} \beta(s) e^{-\int_{s-\tau(s)}^{s} \beta(q)\, dq} ds < \infty.$$

Then

$$\gamma(t) > \left[\exp\left(\int_t^{\infty} \beta(s) e^{-\int_{s-\tau(s)}^{s} \beta(q)\, dq}\, ds \right) - 1 \right] \cdot \xi, \qquad t \in I_1. \quad (15)$$

Proof. From Lemma 5 we have $V(t) < V(t - \tau(t))$, $t \in I_1$ and

$$V'(t) = -\beta(t) e^{-\int_{t_{-1}}^{t} \beta(s)\, ds} y(t - \tau(t)) < -\beta(t) e^{-\int_{t-\tau(t)}^{t} \beta(s)\, ds} V(t). \quad (16)$$

Since $V(\infty) = \xi > 0$ (the constant ξ is the same as in Theorem 5) then, by integration (16) from t to ∞ we have

$$V(t) = \xi + \gamma(t) > \xi \exp\left(\int_t^{\infty} \beta(s) e^{-\int_{s-\tau(s)}^{s} \beta(q)\, dq}\, ds \right), \qquad t \in I_1.$$

From this follows (15). The theorem is proved.

COROLLARY 1. *Let Theorem* 5 *hold, let a solution $y = y_1(t)$ of* (1) *with representation* (11) *be fixed and $\int^{\infty} \beta(s)\, ds = \infty$. Then* (*as it follows from Theorem* 4) *for each solution $y(t)$ of* (1) *on I_{-1} there is a constant $K \in \mathbb{R}$ and a bounded solution $\delta(t)$ of* (1) *such that*

$$y(t) = K y_1(t) + \delta(t) = K(\xi + \gamma(t)) e^{\int_{t_0}^{t} \beta(s)\, ds} + \delta(t), \qquad t \in I_{-1}. \quad (17)$$

If, moreover, Theorem 6 *holds, then* (15) *is valid.*

6. OTHER ASYMPTOTICS

THEOREM 7. *The existence of functions* $\varphi_i \in C([t_\varepsilon, \infty), \mathbb{R}) \cap C^1([t_0 - \varepsilon), \mathbb{R})$, $i = 1, 2$, *where* $\varepsilon \in \mathbb{R}^+$, $\varepsilon = \text{const}$, $t_{-1} + \varepsilon < t_0$, $t_\varepsilon = t_0 - \varepsilon - \tau(t_0 - \varepsilon)$, $\varphi_2(t) \le \varphi_1(t)$, $t \in I_{-1}$ *and*

$$\varphi_1'(t) \le \beta(t)\left[\varphi_1(t) - \varphi_1(t - \tau(t))\right], \qquad t > t_0 - \varepsilon,$$

$$\varphi_2'(t) \ge \beta(t)\left[\varphi_2(t) - \varphi_2(t - \tau(t))\right], \qquad t > t_0 - \varepsilon$$

is sufficient and necessary for existence of solution $y = y(t)$ *of* (1) *such that*

$$\varphi_2(t) \le y(t) \le \varphi_1(t), \qquad t \in I_{-1}. \tag{18}$$

Proof. Necessity is obvious, because $\varphi_1(t) \equiv \varphi_2(t) \equiv y(t)$ can be taken.

Sufficiency. Let us choose an $\delta \ge 0$ such that $\varphi_2(t) < \tilde{\varphi}_1(t) \equiv \varphi_1(t) + \delta$ on I_{-1}. Using topological principle (as in proof of Theorem 3) putting $\omega = \{(t, y) \in \omega_0\colon\ m_1(t, y) < 0,\ l_1(t, y) < 0\}$ where $\omega_0 = (t_\varepsilon, \infty) \times \mathbb{R}$, $m_1(t, y) = t_\varepsilon - t$ and $l_1(t, y) \equiv l_1(y) = (y - \tilde{\varphi}_1(t))(y - \varphi_2(t))$, we can prove that there exists a solution $y = y_\delta(t)$ on I_{-1} (generated, e.g., by an initial function $\Phi_\delta(t, \varepsilon) = \varepsilon(\varphi_1(t) + \delta) + (1 - \varepsilon)\varphi_2(t)$, $t \in [t_{-1}, t_0]$ where $\varepsilon \in (0, 1)$ is a fixed number not depending on δ in further reasonings) which satisfies inequalities,

$$\varphi_2(t) < y_\delta(t) < \tilde{\varphi}_1(t), \qquad t \in I_{-1}.$$

Taking a sequence $\{\delta_n\} \to 0^+$, we conclude that the limit solution $y_0(t) = \lim_{n \to \infty} y_{\delta_n}(t)$, generated by initial function $\Phi_0(t, \varepsilon)$, exists and satisfies (18). If the left inequality of (18) does not hold, then there is $t^\star \in I$ such that $y_0(t^\star) < \varphi_2(t^\star)$. However, this is impossible in view of continuous dependence of solutions on initial data. The case $y_0(t^{\star\star}) > \varphi_1(t^{\star\star})$ for a $t^{\star\star} \in I$ is impossible in view of construction. The theorem is proved. From Theorems 4 and 7 follows

THEOREM 8. *Let there be functions* $\varphi_i(t)$, $i = 1, 2$, *satisfying all conditions described in Theorem* 7 *and, moreover,* $\varphi_1(\infty) = +\infty$, $\lim_{t \to \infty} \varphi_1(t)(\varphi_2(t))^{-1} = 1$. *Then each solution* $y = y(t)$ *of* (1) *can be expressed in the form,*

$$y(t) = K\varphi_i(t)\left[1 + (-1)^i \tilde{\gamma}_i(t)\right] + \tilde{\delta}_i(t), \qquad t \in I_{-1}, i = 1, 2, \tag{19}$$

where $K \in \mathbb{R}$ *is a constant dependent on* $y(t)$; $\tilde{\gamma}_i(t) \ge 0$ *and* $\tilde{\delta}_i(t)$ *are continuous bounded functions dependent on* $y(t)$ *and* $\tilde{\gamma}_i(\infty) = 0$.

EXAMPLE 1. Consider the equation,

$$\dot{y}(t) = \left(\frac{1}{\tau} + \frac{p-1}{2t}\right)\left[y(t) - y(t-\tau)\right], \tag{20}$$

where $p > 1$, $\tau > 0$, $\tau = \text{const}$. By Theorem 8 (suppose that t_0 is sufficiently large) each solution of (20) is representable in the form (19) where $\varphi_1(t) = (t + C)^p$, $C = \text{const}$, $C > \tau(p+1)/6$, $\varphi_2(t) = t^p$. E.g., for $i = 2$ we have

$$y(t) = Kt^p\left[1 + \tilde{\gamma}_2(t)\right] + \tilde{\delta}_2(t), \qquad t \in I_{-1},$$

if t_0 is sufficiently large.

From Theorems 7 and 8 it follows

THEOREM 9. *Let* $\tau(t) \equiv \tau \in \mathbb{R}^+$, $\beta \in C([t_{-1} - \tau + b, \infty), \mathbb{R}^+)$, $a + \tau > b \geq 0$, $a, b \in \mathbb{R}$ *and*

$$\beta(t-b) \leq \frac{\beta(t)}{\int_{t-\tau}^{t}\beta(s)\,ds} \leq \beta(t-a-\tau), \qquad t > t_{-1} + b.$$

If, moreover, $\lim_{t\to\infty}\beta(t-b)/\beta(t-a-\tau) = 1$ *and* $\int^{\infty}\beta(s)\,ds = \infty$, *then each solution* $y = y(t)$ *of* (1) *can be expressed in the form,*

$$y(t) = K\varphi_i(t)\left[1 + (-1)^i\tilde{\gamma}_i(t)\right] + \tilde{\delta}_i(t), \qquad t \in I_{-1},\ i = 1, 2,$$

where $\varphi_2(t) = \int_{t_0-\tau}^{t+b}\beta(s)\,ds$, $\varphi_1(t) = \int_{t_0-\tau}^{t+a+\tau}\beta(s)\,ds$, *and* $K \in \mathbb{R}$ *is a constant dependent on* $y(t)$; $\tilde{\gamma}_i(t) \geq 0$ *and* $\tilde{\delta}_i(t)$ *are continuous bounded functions dependent on* $y(t)$ *and* $\tilde{\gamma}_i(\infty) = 0$.

EXAMPLE 2. Conditions of Theorem 9 are satisfied, e.g., if $\beta(t) = \tau^{-1}(1 + \varepsilon\sin(2\pi/\tau)t)$ where $\varepsilon = \text{const}$, $|\varepsilon| < 1$, and $a = b = 0$. In this case we conclude that each solution $y = y(t)$ of (1) can be expressed in the form,

$$y(t) = Kt\left[1 + (-1)^i\tilde{\gamma}_i(t)\right] + \tilde{\delta}_i(t), \qquad t \in I_{-1},\ i = 1, 2,$$

where $K \in \mathbb{R}$ is a constant dependent on $y(t)$, $\tilde{\gamma}_i(t) \geq 0$, and $\tilde{\delta}_i(t)$ are continuous functions dependent on $y(t)$ and $\tilde{y}_i(\infty) = 0$.

7. APPLICATIONS AND COMPARISONS

(A) In the paper S. N. Zhang [18] there is proved, e.g., the result about asymptotic structure of solutions of (1) under main assumptions $\beta(t) > 0$,

$t > 0$, $\tau(t) \equiv 1$ and

$$\int_t^{t+1} \beta(s)\, ds \geq 1, \qquad \int_t^{t+1} \beta(s)\, ds \not\equiv 1, \tag{21}$$

for $t \geq T_0 > 0$. These conditions are a partial case of condition (i) in Remark 1. Consider, moreover, the equation,

$$\dot{y}(t) = \left(\frac{1}{\tau} - \frac{a}{t}\right)[y(t) - y(t-\tau)], \tag{22}$$

where $\beta(t) = (\tau)^{-1} - at^{-1}$, $\tau(t) \equiv \tau = \text{const}$, $a = \text{const}$, and $a \in (0, \frac{1}{2}]$. Then neither conditions (21) for $\tau = 1$ nor condition (i) in Remark 1 hold. In this case the condition (iii) in Remark 1 holds (if t_{-1} is sufficiently large), $\omega(+\infty) = +\infty$ and therefore there exists a nonbounded positive solution $y(t)$ of (1) which tends to infinity and satisfies inequality $y(t) > a \ln t/t_{-1}$. Clearly, Theorem 4 holds. This examples shows simultaneously that the value $a = \frac{1}{2}$ is a bifurcation value. Indeed, in the paper of F. V. Atkinson and J. R. Haddock [2] it is shown that in the case $a > \frac{1}{2}$ each solution of (22) tends to a constant.

(B) The equation of the type (2) where $c \in C(I_{-1}, \mathbb{R}^+)$ can be obtained from equation (1) by transformation $y(t) = x(t)\exp(\int_{t_0}^t \beta(s)\, ds)$. Conversely, if there is a positive solution of (2) $x(t) = \zeta(t)$ on I_{-1}, then the transformation $x(t) = \zeta(t)y(t)$ gives

$$\dot{y}(t) = -\frac{\zeta'(t)}{\zeta(t)}[y(t) - y(t - \tau(t))], \qquad t \in I_{-1}, \tag{23}$$

that is (23) has the form of (1) with $\beta(t) \equiv -\zeta'(t)/\zeta(t)$. Note that sufficient and necessary conditions for existence of a positive solution of (2) are given by Detang Zhou [19] (see L. H. Erbe, Qingkai Kong, and B. G. Zhang [6] too).

THEOREM 10. *Suppose* $c \in C([t_\varepsilon, \infty), \mathbb{R}^+)$, *where* $\varepsilon \in \mathbb{R}^+$, $\varepsilon = \text{const}$, $t_{-1} + \varepsilon < t_0$, $t_\varepsilon = t_0 - \varepsilon - \tau(t_0 - \varepsilon)$, $c(t) < 1$ *on* I_{-1}. *Let, moreover, there be a positive constant L such that*

$$c(t) \leq L\left(\ln \frac{1}{c(t)}\right)\exp\left(L\int_{t-\tau(t)}^t \ln c(s)\, ds\right), \qquad t > t_0 - \varepsilon.$$

Then there is a positive solution of (2) $x = \zeta(t)$ *such that*

$$0 < \zeta(t) < \exp\left(L\int_{t_0}^t \ln c(s)\, ds\right), \qquad t \in I_{-1}. \tag{24}$$

Proof. The existence of a positive solution of (2) $x = \zeta(t)$ with property (24) can be proved by means of topological principle analogously as in the proof of Theorem 3. For this it is necessary to choose $\omega = \{(t, y) \in \omega_0 : m_1(t, y) < 0,\ l_1(t, y) < 0\}$ where $\omega_0 = (t_\varepsilon, \infty) \times \mathbb{R}$, $m_1(t, y) = t_\varepsilon - t$, and $l_1(t, x) \equiv l_1(x) = x \cdot (x - \exp(L \int_{t_0}^{t} \ln c(s)\, ds))$. The details are omitted.

THEOREM 11. *Let Theorem* 10 *hold. Suppose, moreover, that there is a constant* $p \in (0, 1]$ *such that* $\int^\infty c^p(s)\, ds < \infty$, $\int^\infty \ln c(s)\, ds = -\infty$ *and*

$$c^{p-1}(t) \geq \exp\left(\int_{t-\tau(t)}^{t} c^p(s)\, ds \right), \qquad t \in I_{-1}. \tag{25}$$

Then for each solution of (2) $x = x(t)$ *there is a constant* $K \in \mathbb{R}$ *and a bounded solution of* (23) $\delta(t)$ *such that*

$$x(t) = K\zeta(t_0)(\xi + \gamma(t)) + \zeta(t)\delta(t), \qquad t \in I_{-1}, \tag{26}$$

where $\zeta(t)$ *is the positive solution of* (2) *given by Theorem* 10, $\xi > 0$ *is a fixed constant and* $\gamma(t) \geq 0$ *is a fixed continuous function with property* $\gamma(\infty) = 0$.

$$\int_{t_0}^{\infty} \beta(s)\, ds = -\int_{t_0}^{\infty} \frac{\zeta'(s)}{\zeta(s)}\, ds = -\ln \frac{\zeta(+\infty)}{\zeta(t_0)} = +\infty.$$

Put $\alpha(t) = -c^p(t)\zeta(t)/\zeta'(t)$. Without loss of generality we suppose that $\zeta \in C^1(I_{-1}, \mathbb{R}^+)$. We see that $\int^\infty \alpha(s)\beta(s)\, ds = \int^\infty c^p(s)\, ds < \infty$ and (12) holds in view of (25). Therefore, by Corollary 1 for each solution of (23), there is a constant $\xi \in \mathbb{R}$ and a bounded function $\delta(t)$ continuous on I_{-1} such that representation (17) holds and, consequently, for each solution of (2) there is a constant $\xi \in \mathbb{R}$ and a bounded function $\delta(t)$ continuous on I_{-1} such that

$$x(t) = K\zeta(t_0)(\xi + \gamma(t)) + \zeta(t)\delta(t).$$

The theorem is proved.

The proof of the following theorem is similar to that of Theorem 11 and hence is omitted.

THEOREM 12. *Let there be a positive solution of* (2) $x(t) = \zeta(t)$ *on* I_{-1} *such that* $\zeta(\infty) = 0$ *and* $c \in C([t_\varepsilon, \infty), \mathbb{R}^+)$. *Assume, moreover, that there is a constant* $p \in (0, 1]$ *such that* $\int^\infty c^p(s)\, ds < \infty$ *and that* (25) *holds. Then for each solution of* (2) $x = x(t)$ *there is a constant* $K \in \mathbb{R}$ *and a bounded solution of* (23) $\delta(t)$ *such that the representation* (26) *holds where* $\xi > 0$ *is a fixed constant and* $\gamma(t) \geq 0$ *is a fixed continuous function with property* $\gamma(\infty) = 0$.

Remark 4. The existence of a positive solution (without guarantee that its asymptotic behavior is described by inequalities (24)) follows from (25) because previously mentioned sufficient and necessary conditions are valid (see [19, 6]). Note, moreover, that, by means of appropriate transformation of variables, it is possible, under our assumptions, to obtain equations with *constant* delay that are equivalent to (1) and (2). This follows from results given in [13, 14]. If Theorems 11 or 12 hold, then (as it follows from formula (26)) the trivial solution of (2) is *nonasymptotically* stable on I_{-1} because by Remark 2 there is a solution of (2) for each $K \neq 0$ and therefore for $K \neq 0$ $\lim_{t \to \infty} x(t) = K\zeta(t_0)\xi \neq 0$.

EXAMPLE 3. Consider the equation (see, e.g., J. Hale [10]),

$$\dot{x}(t) = -2te^{1-2t}x(t-1).$$

Because it follows from Theorems 11 and 12, each solution of this equation is representable in the form (26). Because there is a positive solution $\zeta(t) = \exp(-t^2)$, this representation can be written in the form,

$$x(t) = Ke^{-t_0^2}(\xi + \gamma(t)) + e^{-t^2}\delta(t), \qquad t \in I_{-1}.$$

Moreover, by Remark 4, the trivial solution of this equation is nonasymptotically stable.

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